<u>All Highly Abundant Numbers > 10 Are Practical</u>

A positive integer N is *highly abundant* when its sum of divisors function $\sigma(N) > \sigma(M)$ for all positive integers $M < N$. N is said to be *practical* when every positive integer less than or equal to N is a sum of distinct divisors of N.

In 2013, on the *Online Encyclopedia of Integer Sequences* page for 'highly abundant numbers' (A 002093) J.Coleman conjectured that all highly abundant numbers > 10 are practical and that he had verified his conjecture for the first 10,000 highly abundant numbers (or, in other words, for all highly abundant numbers to $7.4 * 10^{27}$).

The object of this note is to prove Coleman's conjecture. We use only elementary methods. The main part of the proof is in the

<u>Lemma:</u> *If N is not practical and* $6|\ N$*, then N is not highly abundant.*

<u>Proof:</u> Using the Stewart- Sierpinski criterion for N to be practical in terms of its prime factorizations (see [ ]) we can say that

$N = M R$ where $6|\ M$ and every prime factor p of R is larger than $\sigma(M)+1$. Let p be the smallest prime factor of R. We shall show N is not highly abundant by producing a number N_1 which is smaller than N with

$\sigma(N) < \sigma(N_1)$. Let N_1 be $M\,(R/p)\,(p\text{-}1)$. By adding up some of the divisors of N_1 and getting a sum greater than $\sigma(N)$, we shall prove the lemma. In particular, $n' = (p-1)/2$ and $3\,n'$ are divisors of N_1. The following sets of numbers are all divisors of N_1:

1) Divisors of M (R/p) multiplied by 1 and (p-1)
2) Divisors of R/p multiplied by n'
3) Divisors of R/p multiplied by $3n'$.

Example (part 1): Let $N = 1938 = (2)(3)(17)(19)$. Then $M = 6$ and $R = 323 = (17)(19)$. Likewise $p = 17$ and $n' = 8$. Then the following are the sets of divisors given by 1), 2) and 3), of $N_1 = (2)(3)(16)(19) = 1824$

1) {1, 2, 3, 6, 19, 38, 57, 114, 16, 32, 48, 96, 304, 608, 912, 1824}
2) { 8,152}
3) {24,456}

Most importantly , these sets are all disjoint , Since prime divisors of R are all greater than $\sigma(M)+1$ and $6|\, M$, they are all at least 17. So 3 does not divide R and sets 2) and 3) are disjoint. Since any divisor of R/p is odd, sets 2) and 3) consist of odd multiples of n'. Divisors in set 1) are either multiples of $p\text{-}1 = 2n'$ and not in 2) or 3) or they are a divisor of MR/p. But n' is smaller than any prime divisor of R and so is relatively prime to it. Also, since $p > \sigma(M)+1$ and $6|\, M$, we have that $\sigma(M) \geq 2M$.

Therefore $p > 2M + 1$ and $n' > M$. So no multiple of n' is a divisor of M and set 1) is disjoint from sets 2) and 3).

So $\sigma(N_1)$ is greater than or equal to $S_1 + S_2 + S_3$, the sums of the divisors from the sets 1),2), and 3), respectively. What are the sums?

The divisors on set 1) can also be obtained in this way. Take the set of divisors of N, our original number. If a divisor is not divisible by p, leave it alone and if p divides it, multiply the divisor by $(p\text{-}1)/p$. This gives the divisors in set 1) and tells us that their sum is $S_1 = \sigma(N) - \sigma(MR/p)$.

Straightforwardly, $S_2 = n' \sigma(R/p)$ and $S_3 = 3\, n' \sigma(R/p)$.

So finally we have $\sigma(N_1) \geq \sigma(N) - \sigma(MR/p) + n' \sigma(R/p) + 3\, n' \sigma(R/p) =$

$\sigma(N) - \sigma(M)\, \sigma(R/p) + 4\, n' \sigma(R/p) = \sigma(N) + (4n' - \sigma(M))\, \sigma(R/p)$. As before, $p\text{-}1 = 2\, n' > \sigma(M)$ so $4\, n' - \sigma(M) > 0$ and $\sigma(N_1) > \sigma(N)$. Q.E.D.

Example (part 2): Now $\sigma(N) = (3)(4)(18)(20) = 4320$. The sums of the divisors S_1, S_2, S_3 are $S_1 = 4320 - \sigma(114) = 4320 - 240 = 4080$, $S_2 = 160$, and $S_3 = 480$. So $\sigma(N_1)$ is at least $4080 + 160 + 480 = 4720 > 4320$. (In fact, $\sigma(N_1) = (63)(4)(20) = 5040$ because of the divisors not counted by any of the sets.)

Theorem: *If N is a highly abundant number greater than* 10, *then N is practical.*

Proof: D.Fischer has proven that highly abundant numbers > 20 are divisible by 6, and so they are all practical by the lemma. The highly abundant numbers greater than 10 and less than or equal 20 are 12, 16,18, and 20 which are all practical. Q.E.D.

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